

ON THE SYMMETRIC ENVELOPING ALGEBRA OF PLANAR ALGEBRA SUBFACTORS

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ABSTRACT. We give a diagrammatic description of Popa's symmetric enveloping algebras associated to planar algebra subfactors. As an application we construct a natural family of derivations on these factors, and compute a certain free entropy dimension type quantity.

INTRODUCTION

Inspired by Voiculescu's [19] description of the large N limit of the distribution of families of independent random matrices, Guionnet, Jones and Shlyakhtenko [7] introduced diagrammatic traces on a tower of graded algebras $Gr_0(\mathcal{P}) \subset Gr_1(\mathcal{P}) \subset \cdots$ which is naturally associated to a subfactor planar algebra \mathcal{P} . The main result of [7] is that the completions form a tower of II_1 factors $M_0 \subset M_1 \subset \cdots$ whose standard invariant is \mathcal{P} , thus giving a diagrammatic proof of a fundamental result of Popa [15]. In subsequent work [8] they were able to identify the isomorphism class of these factors in the case that \mathcal{P} is finite-depth. Explicitly, M_k is isomorphic to an interpolated free group factor $L(\mathbb{F}_{r_k})$, where

$$(*) \quad r_k = 1 + 2\delta^{-2k} I(\delta - 1).$$

Here δ^2 is the index $[M_1 : M_0]$, see Section 3 for the definition of the global index I . The fact that M_k is an interpolated free group factor was also obtained independently by Kodiyalam and Sunder [12]. See also [9],[10],[11].

Given an inclusion of II_1 factors $N \subset M$, Popa [14] has shown that there is a unique II_1 factor $M \boxtimes_{e_N} M^{op}$, called the *symmetric enveloping algebra* of the inclusion, which is generated by a copy of $M \otimes M^{op}$ and a projection e_N which is simultaneously the Jones projection for the inclusions $N \subset M$ and $N^{op} \subset M^{op}$. This algebra, together with the inclusion $M \otimes M^{op} \subset M \boxtimes_{e_N} M^{op}$, encodes a number of important analytic properties of the subfactor $N \subset M$, such as amenability and property T [16]. When the inclusion $N \subset M$ is *strongly amenable*, for example if M is hyperfinite and $N \subset M$ is finite-depth, then $M \otimes M^{op} \subset M \boxtimes_{e_N} M^{op}$ is isomorphic to Ocneanu's *asymptotic inclusion* [13].

In this paper we give a diagrammatic construction of the symmetric enveloping algebras of the inclusions $M_k \subset M_{k+1}$. (In the case that \mathcal{P} is the Temperley-Lieb planar algebra, this was announced in [17]). Using this description, we give a simple diagrammatic proof that the index $[M_k \boxtimes_{e_{k-1}} M_k^{op} : M_k \otimes M_k^{op}]$ is the global index I , first computed by Ocneanu [13]. As an application we construct a module of closable derivations $\delta : Gr_0 \rightarrow Gr_0 \otimes Gr_0^{op}$, related to the Schwinger-Dyson equation from [9], whose Murray von-Neumann dimension (after completion) is precisely the parameter r_0 from (*). Since the dimensions of such spaces of derivations are well-known to be related to Voiculescu's free entropy dimension (see e.g. [18]), this provides a more conceptual understanding of the rather mysterious parameters r_k computed in [8].

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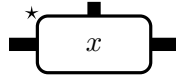
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Our paper is organized as follows. We begin by briefly recalling the construction from [7] in Section 1. In Section 2 we construct an auxiliary semi-finite algebra \mathfrak{M} , starting from a subfactor planar algebra \mathcal{P} . Section 3 contains our main result: the symmetric enveloping algebras of the inclusions $M_k \subset M_{k+1}$ are naturally isomorphic to certain compressions of \mathfrak{M} . We then compute the index of $M_k \otimes M_k^{op} \subset M_k \boxtimes_{e_{k-1}} M_k^{op}$. In Section 4 we construct derivations on Gr_0 , and compute a free entropy dimension type quantity for M_0 .

1. PLANAR ALGEBRA SUBFACTORS

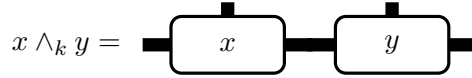
In this section we briefly recall the construction from [7]. The reader is referred to this paper and to [10], [8] for further details.

Let \mathcal{P} be a subfactor planar algebra with graded components P_k^ϵ , $k = 0, 1, 2, \dots$, $\epsilon = \pm$. For $n, k \geq 0$ and $\epsilon = \pm$ let $P_{n,k}^\epsilon$ be a copy of P_{n+k}^ϵ . Elements of $P_{n,k}^\epsilon$ will be represented by diagrams

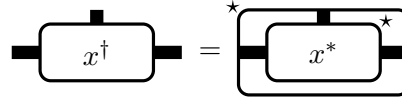


where the thick lines to the left and right represent k strings, and the thick line at top represents $2n$ strings. We will typically suppress the marked point \star , and take the convention that it occurs at the top-left corner and is shaded according to ϵ .

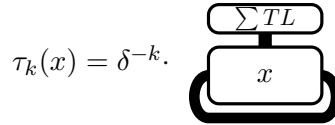
Define a product $\wedge_k : P_{n,k}^\epsilon \times P_{m,k}^\epsilon \rightarrow P_{n+m,k}^\epsilon$ by



The involution $\dagger : P_{n,k}^\epsilon \rightarrow P_{n,k}^\epsilon$ is given by

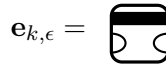


The Voiculescu trace $\tau_k : P_{n,k}^\epsilon \rightarrow \mathbb{C}$ is defined by

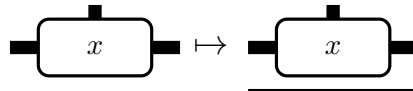


where $\sum TL$ denotes the sum over all loopless Temperley-Lieb diagram with $2n$ boundary points. Let $Gr_k^\epsilon(\mathcal{P}) = \bigoplus_{n \geq 0} P_{n,k}^\epsilon$, and observe that the formulas above give $Gr_k^\epsilon(\mathcal{P})$ the structure of a graded $*$ -algebra with trace τ_k . The unit of $Gr_k^\epsilon(\mathcal{P})$ is the element of $P_{0,k}^\epsilon$ consisting of k parallel lines.

Let $e_{k,\epsilon}$ denote the following element of $P_{0,k+2}^\epsilon$, $k \geq 0$:



Note that there are natural inclusions of $Gr_k^\epsilon(\mathcal{P})$ into $Gr_{k+1}^\epsilon(\mathcal{P})$ defined by



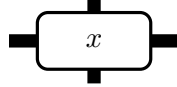
The main result of [7] is the following:

Theorem 1.1. *For $k \geq 0$ and $\epsilon = \pm$, the Voiculescu trace τ_k is a faithful tracial state on $Gr_k^\epsilon(\mathcal{P})$, and its GNS completion is a II_1 factor M_k^ϵ as long as $\delta > 1$. The inclusions $Gr_k^\epsilon(\mathcal{P}) \subset Gr_{k+1}^\epsilon(\mathcal{P})$ extend to $M_k^\epsilon \subset M_{k+1}^\epsilon$, and $(M_{k+1}^\epsilon, \mathbf{e}_{k,\epsilon})$ is the tower of the basic construction for the subfactor $M_0^\epsilon \subset M_1^\epsilon$. Moreover, the planar algebra of $M_0 \subset M_1$ is isomorphic to \mathcal{P} .*

□

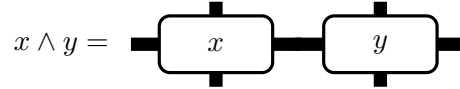
2. A SEMI-FINITE ENVELOPING ALGEBRA ASSOCIATED TO A PLANAR ALGEBRA

Let \mathcal{P} be a subfactor planar algebra. For integers k_1, k_2, s, t which add to $2n$, and $\epsilon = \pm$, let $V_{k_1, k_2}^\epsilon(s, t)$ be a copy of P_n^ϵ . Elements of $V_{k_1, k_2}^\epsilon(s, t)$ will be represented by diagrams of the form

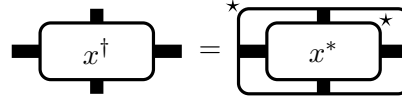


where the thick lines at left, right, top and bottom represent k_1, k_2, s and t strings, respectively. We take the convention that the marked point occurs at the upper left corner, which is shaded ϵ . If ϵ is not specified we assume that $\epsilon = +$.

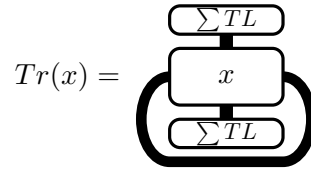
Define a product $\wedge : V_{k_1, k_2}^\epsilon(s, t) \times V_{k'_1, k'_2}^{\epsilon'}(s', t') \rightarrow V_{k_1, k'_2}^\epsilon(s + s', t + t')$ to be zero unless $k_2 = k'_1$ and $\epsilon' = (-1)^s \cdot \epsilon$, and otherwise by



The adjoint $\dagger : V_{k_1, k_2}^\epsilon(s, t) \rightarrow V_{k_2, k_1}^{(-1)^s \epsilon}(s, t)$ is defined by



Define $Tr : V_{k_1, k_2}^\epsilon(s, t) \rightarrow \mathbb{C}$ to be zero unless $k_1 = k_2$ and both s and t are even, and otherwise by



Let

$$V = \bigoplus_{\epsilon = \pm} \bigoplus_{\substack{k_1, k_2, s, t \geq 0 \\ k_1 + k_2 + s + t \text{ even}}} V_{k_1, k_2}^\epsilon(s, t).$$

It is easy to see that the formulas above give V the structure of a (non-unital) graded $*$ -algebra with trace Tr .

Theorem 2.1. *Tr is a faithful, positive linear functional on V . Moreover, V acts by bounded operators on the GNS Hilbert space for Tr .*

The (semi-finite) von Neumann algebra obtained by completing V under the GNS representation for Tr will be denoted \mathfrak{M} . To prove Theorem 2.1 we use the orthogonalization method of [10].

2.1. An orthogonal basis. We will now define a new multiplication and trace on V . To distinguish between these two structures, we define $W_{k_1, k_2}^\epsilon(s, t)$ to be a copy of $V_{k_1, k_2}^\epsilon(s, t)$, and let W be the same direct sum defining V .

Define a product \star on $W_{k_1, k_2}^\epsilon(s, t) \times W_{k'_1, k'_2}^{\epsilon'}(s', t')$ to be zero unless $k_2 = k'_1$ and $\epsilon' = (-1)^s \cdot \epsilon$, and otherwise by

$$x \star y = \sum_{\substack{1 \leq i \leq \min(s, s') \\ 1 \leq j \leq \min(t, t')}} \text{Diagram}$$

where i and j mean that there are i parallel strings at top and j parallel strings at bottom. The numbers of the other parallel strings are then determined by our conventions.

Define a trace $Tr' : W_{k_1, k_2}^\epsilon(s, t)$ to be zero unless $k_1 = k_2$ and $s = t = 0$, and otherwise by

$$Tr'(x) = \text{Diagram}$$

Observe that there is a natural action of $TL \otimes TL^{op}$ on V as follows: if a, b are TL diagrams with s' (resp. t) points on top and s (resp. t') points on bottom, then $a \otimes b$ acts as the sum over k_1, k_2 and ϵ of the linear maps $V_{k_1, k_2}^\epsilon(s, t) \rightarrow V_{k_1, k_2}^\epsilon(s', t')$ defined by

$$x \mapsto \text{Diagram}$$

Recall that a TL diagram is called *epi* if each point on the top is connected to a point on the bottom, and *monic* if each point on the bottom is connected to a point on top. Furthermore, an epi (resp. monic) TL diagram is said to be *nonnested* if each “turn-back” on the bottom (resp. top) of the diagram encloses no other turn-backs.

Define $X : V \rightarrow W$ to be the sum of all $a \otimes b$ where a (resp. b) are epi (resp. monic) TL diagrams. Likewise, define $Y : W \rightarrow V$ to be the sum of all $a \otimes b$ where a (resp. b) are nonnested epi (resp. monic) TL diagrams, and the coefficient on $a \otimes b$ is $(-1)^{s+t-s'-t'}$ when a, b are as above.

Proposition 2.2.

- (1) $XY = 1 = YX$.
- (2) $X(a \wedge b) = X(a) \star X(b)$.
- (3) $X(a^*) = X(a)^*$.
- (4) $Tr'(X(a)) = Tr(a)$.

Proof. The proof is identical to the argument in [10, Section 5], and will therefore be omitted. \square

Corollary 2.3. $(W, \star, *)$ is an associative $*$ -algebra.

Proof. Since $(V, \wedge, *)$ is clearly associative, this follows from the isomorphism in Proposition 2.2. \square

We are now prepared to prove Theorem 2.1.

Proof of Theorem 2.1. By Proposition 2.2, to prove the theorem it suffices to show that Tr' is a faithful, positive state on W and that W acts by bounded operators on the GNS Hilbert space for Tr' . That Tr' is positive and faithful is one of the defining axioms of the subfactor planar algebra \mathcal{P} . The fact that W acts by bounded operators on the associated GNS Hilbert space may be proved along the same lines as [10, Theorem 3.3]. \square

then $\mathbf{f}_{k,+}$ implements the Jones projections for both $M_{k-1} \subset M_k$ and $M_{k-1}^{op} \subset M_k^{op}$. It now follows from [14, Theorem 1.2] and the discussion above that $W^*(M_k \otimes M_k^{op}, \mathbf{f}_k)$ is isomorphic to

Let $k \geq 0$ and $\epsilon = \pm$, and define $c_{k,\epsilon} \in V_{2k,2k+2}^\epsilon(1,1)$ by

$$c_{k,\epsilon} = \text{Diagram of a box with a horizontal band and a circle containing } 2k$$

It follows from [7, Lemma 2] that $c_{k,\epsilon}c_{k,\epsilon}^*$ is invertible in $M_k^\epsilon \boxtimes M_k^\epsilon$. Let $q_{k,\epsilon}$ be the initial projection of $c_{k,\epsilon}^*c_{k,\epsilon}$,

$$q_{k,\epsilon} = c_{k,\epsilon}^* (c_{k,\epsilon} c_{k,\epsilon}^*)^{-1} c_{k,\epsilon}.$$

Lemma 3.2. *Fix $k \geq 0$ and $\epsilon = \pm$.*

(1) *The map*

$$\theta_{k,\epsilon}(x) = c_{k,\epsilon}^*(c_{k,\epsilon}c_{k,\epsilon}^*)^{-1/2}x(c_{k,\epsilon}c_{k,\epsilon}^*)^{-1/2}c_{k,\epsilon}$$

is an isomorphism of $M_k^\epsilon \boxtimes M_k^\epsilon$ onto $q_{k,\epsilon}(M_{k+1}^{-\epsilon} \boxtimes M_{k+1}^{-\epsilon})q_{k,\epsilon}$.

(2) If $k \geq 1$, then $\theta_{k,\epsilon}$ restricts to an isomorphism

$$W^*(M_k^\epsilon \otimes (M_k^\epsilon)^{op}, \mathbf{f}_{k,\epsilon}) \simeq q_{k,\epsilon}(W^*(M_{k+1}^{-\epsilon} \otimes (M_{k+1}^{-\epsilon})^{op}, \mathbf{f}_{k+1,-\epsilon}))q_{k,\epsilon}.$$

Proof. (1) is straightforward, the inverse of $\theta_{k,\epsilon}$ is given by

$$y \mapsto (c_{k,\epsilon} c_{k,\epsilon}^*)^{-1/2} c_{k,\epsilon} y c_{k,\epsilon}^* (c_{k,\epsilon} c_{k,\epsilon}^*)^{-1/2}.$$

For (2), since it is clear that $\theta_{k,\epsilon}$ takes $M_k^\epsilon \otimes (M_k^\epsilon)^{op}$ onto $q_{k,\epsilon}(M_{k+1}^{-\epsilon} \otimes (M_{k+1}^{-\epsilon})^{op})q_{k,\epsilon}$ and $\mathbf{f}_{k,\epsilon}$ to $q_{k,\epsilon}\mathbf{f}_{k+1,-\epsilon}q_{k,\epsilon}$, it suffices to show that $q_{k,\epsilon}W^*(M_{k+1}^{-\epsilon} \otimes (M_{k+1}^{-\epsilon})^{op}, \mathbf{f}_{k+1,-\epsilon})q_{k,\epsilon}$ is generated by $q_k(M_{k+1}^{-\epsilon} \otimes (M_{k+1}^{-\epsilon})^{op})q_{k,\epsilon}$ and $q_{k,\epsilon}\mathbf{f}_{k+1,-\epsilon}q_{k,\epsilon}$. Note that $q_{k,\epsilon}$ is contained in the II_1 factor $M_1^\epsilon \otimes (M_1^\epsilon)^{op}$, so there are partial isometries $v_1, \dots, v_n \in M_1^\epsilon \otimes (M_1^\epsilon)^{op}$ such that $v_i^*v_i \leq q_{k,\epsilon}$ and $\sum_i v_i^*q_{k,\epsilon}v_i = 1$. Let $m_1, \dots, m_l \in M_{k+1}^\epsilon \otimes (M_{k+1}^{-\epsilon})^{op}$, then

$$\begin{aligned}
& q_{k,\epsilon}(m_1 \mathbf{f}_{k+1,-\epsilon} m_2 \cdots \mathbf{f}_{k+1,-\epsilon} m_l) q_{k,\epsilon} \\
&= q_{k,\epsilon} m_1 \left(\sum_{i_1=1}^n v_{i_1}^* q_{k,\epsilon} v_{i_1} \right) \mathbf{f}_{k+1,-\epsilon} m_2 \left(\sum_{i_2=1}^n v_{i_2}^* q_{k,\epsilon} v_{i_2} \right) \cdots m_{l-1} \left(\sum_{i_{l-1}=1}^n v_{i_{l-1}}^* q_{k,\epsilon} v_{i_{l-1}} \right) \mathbf{f}_{k+1,-\epsilon} m_l q_{k,\epsilon} \\
&= \sum_{1 \leq i_1, \dots, i_{l-1} \leq n} q_{k,\epsilon} (m_1 v_{i_1}^*) q_{k,\epsilon} \mathbf{f}_{k+1,-\epsilon} q_{k,\epsilon} (v_{i_1} m_2 v_{i_2}^*) q_{k,\epsilon} \mathbf{f}_{k+1,-\epsilon} \cdots \mathbf{f}_{k+1,-\epsilon} q_{k,\epsilon} (v_{i_{l-1}} m_l) q_{k,\epsilon},
\end{aligned}$$

where we have used the fact that $\mathbf{f}_{k+1,-\epsilon}$ commutes with $M_1^{-\epsilon} \otimes (M_1^{-\epsilon})^{op}$. The result follows. \square

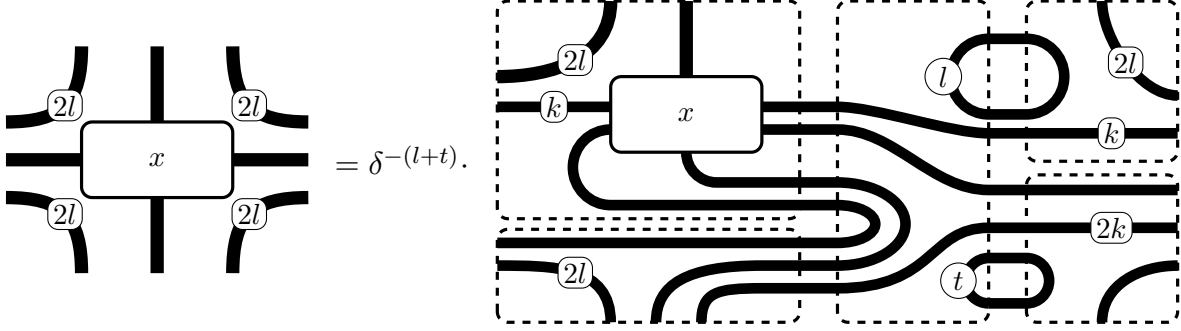
Theorem 3.3. *For any $k \geq 1$ we have*

$$M_k \boxtimes M_k \simeq M_k \boxtimes_{e_{k-1}} M_k^{op}.$$

Proof. As discussed above, we just need to show that $M_k \boxtimes M_k = W^*(M_k \otimes M_k^{op}, \mathbf{f}_k)$. If $x \in Gr_k(\mathcal{P}) \boxtimes Gr_k(\mathcal{P})$, then by Lemma 3.2 we have $x \in W^*(M_k \otimes M_k^{op}, \mathbf{f}_k)$ if $c_{k,+}^* x c_{k,+} \in W^*(M_{k+1}^- \otimes (M_{k+1}^-)^{op}, \mathbf{f}_{k+1}, -)$. Iterating, it is enough to show that for l sufficiently large we have

$$\begin{array}{c} \textcircled{2l} \\ | \\ \text{---} x \text{---} \\ | \\ \textcircled{2l} \end{array} \in W^*(M_{k+2l} \otimes M_{k+2l}^{op}, \mathbf{f}_{k+2l})$$

Let $x \in V_{2k,2k}(2s, 2t)$, then for $l = t + k$ we have



is the product of three elements in $M_{k+2l} \boxtimes M_{k+2l}$. The first and third terms are (manifestly) elements of $M_{k+2l} \otimes M_{k+2l}^{op}$. The second term is a Temperley-Lieb element, but since $W^*(M_{k+2l} \otimes M_{k+2l}^{op}, \mathbf{f}_{k+2l})$ contains all of the standard Temperley-Lieb generators

$$e_i = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad (i = 0, \dots, 2k + 4l - 2),$$

it contains any such Temperley-Lieb element. This completes the proof. \square

We will now give a diagrammatic computation of the index $[M_0 \boxtimes M_0 : M_0 \otimes M_0^{op}]$. Let Γ be the principal graph of \mathcal{P} (see e.g. [6]), and let Γ_+ denote the collection of even vertices. Associated to each $v \in \Gamma_+$ is an irreducible $M_0 - M_0$ bimodule X_v . Let $*$ denote the distinguished vertex, corresponding to $X_* = L^2(M_0)$, and for $k \geq 0$ let $\Gamma_{+,k}$ denote the set of vertices in Γ whose distance from $*$ is less than or equal to $2k$. Then we have

$$M_0 L^2(M_k)_{M_0} \simeq \bigoplus_{v \in \Gamma_{+,k}} X_v \otimes \mathcal{H}_v(k),$$

where $\mathcal{H}_v(k)$ are auxiliary finite-dimensional Hilbert spaces, whose dimensions we denote by $n_k(v)$.

It follows that

$$M_0 \otimes M_0^{op} L^2(M_k \otimes M_k^{op})_{M_0 \otimes M_0^{op}} \simeq \bigoplus_{v,w \in \Gamma_{+,k}} X_v \otimes \overline{X_w} \otimes (\mathcal{H}_v(k) \otimes \overline{\mathcal{H}_w(k)}),$$

where $\overline{X_w}$ is the contragredient bimodule and $\overline{\mathcal{H}_w(k)}$ is the conjugate Hilbert space. Let $1_v(k) \in \mathcal{H}_v(k) \otimes \overline{\mathcal{H}_v(k)}$ be the unit under the natural identification with $\text{Hom}_{\mathbb{C}}(\mathcal{H}_k(v))$, and define $p_k(v)$ to be the projection from $L^2(M_k \otimes M_k^{op})$ onto $X_v \otimes \overline{X_v} \otimes 1_v(k)$. Then define

$$p_k = \sum_{v \in \Gamma_{+,k}} p_k(v).$$

Recall that P_{2k} can be identified with $\text{Hom}_{M_0, M_0}(L^2(M_k))$, in particular the central projections $q_k(v)$ of P_{2k} are indexed by $v \in \Gamma_{+,k}$. Let $t_k(v)$ denote the trace of a minimal projection in $P_{2k}(v) := P_{2k} \cdot q_k(v)$, the central component of P_{2k} corresponding to $v \in \Gamma_{+,k}$. If $\{e_{ij}(v) : 1 \leq i, j \leq n_k(v)\}$ is a choice of matrix units for $P_{2k}(v)$, then we have

$$p_k(v) = \frac{1}{n_k(v)} \sum_{1 \leq i, j \leq n_k(v)} e_{ij}(v) \otimes e_{ji}(v)^{op}.$$

Graphically, we will represent $p_k(v)$ using a Sweedler type convention as follows:

$$p_k(v) = \begin{array}{c} \boxed{p_k^{(1)}(v)} \\ \boxed{p_k^{(2)}(v)} \end{array}$$

Lemma 3.4. *Let $x, y \in P_{2k}$ and $v \in \Gamma_{+,k}$, then we have*

$$\begin{array}{c} \text{---} \boxed{p_k^{(1)}(v)} \text{---} x \text{---} \\ \text{---} \boxed{p_k^{(2)}(v)} \text{---} y \text{---} \end{array} = \frac{\delta^{2k} \cdot t_k(v)}{n_k(v)} \cdot \begin{array}{c} x \\ y \end{array} \begin{array}{c} \boxed{q_k(v)} \end{array} \star$$

Proof. Let Tr_v denote the unnormalized trace on $P_{2k}(v) \simeq M_{n_k(v)}(\mathbb{C})$, so that $\tau_k|_{P_{2k}(v)} = t_k(v) \cdot Tr_v$. Then the left hand side of the formula in the statement is equal to

$$\begin{aligned} \frac{1}{n_k(v)} \sum_{1 \leq i, j \leq n_k(v)} \delta^{4k} \cdot t_v(k)^2 Tr_v(e_{ij}(v)x) Tr_v(y e_{ji}(v)) \\ = \frac{\delta^{2k} \cdot t_k(v)}{n_k(v)} \sum_{1 \leq i, j \leq n_k(v)} \delta^{2k} \cdot t_k(v) Tr_v(xy q_k(v)), \end{aligned}$$

which is equal to the right hand side. \square

Define $\Psi_k : Gr_k \otimes Gr_k^{op} \rightarrow Gr_0 \boxtimes Gr_0$ by linear extension of

$$\Psi_k(x \otimes y^{op}) = \sum_{v \in \Gamma_{+,k}} \sqrt{\frac{t_k(v)}{n_k(v)}} \cdot \begin{array}{c} x \\ q_k(v) \\ y \end{array} \star$$

Theorem 3.5. Ψ_k extends to a $M_0 \otimes M_0^{op}$ -bilinear partial isometry from $L^2(M_k \otimes M_k^{op})$ into $L^2(M_0 \boxtimes M_0)$, whose initial projection is equal to p_k .

Proof. Let $x_1, x_2, y_1, y_2 \in Gr_k$, then $\langle \Psi_k(x_1 \otimes y_1^{op}), \Psi_k(x_2 \otimes y_2^{op}) \rangle$ is equal to

$$\sum_{v, w \in \Gamma_{+,k}} \sqrt{\frac{t_k(v)t_w(k)}{n_k(v)n_w(k)}} \begin{array}{c} \boxed{\sum TL} \\ x_1 \quad x_2^* \\ q_k(v) \quad q_w(k) \\ y_1 \quad y_2^* \\ \boxed{\sum TL} \end{array}$$

Let x, y^{op} denote the elements of P_{2k}, P_{2k}^{op} represented by the dashed boxes above. Since $q_k(v)$ and $q_w(k)$ are central, the terms in the sum above for which $v \neq w$ are equal to zero. But applying

Lemma 3.4 to x, y , we see that the term corresponding to $v = w$ is equal to $\langle p_k(v) \cdot (x_1 \otimes y_1^{op}), x_2 \otimes y_2^{op} \rangle$. So we have proved that

$$\langle \Psi_k(x_1 \otimes y_1^{op}), \Psi_k(x_2 \otimes y_2^{op}) \rangle = \langle p_k \cdot (x_1 \otimes y_1^{op}), x_2 \otimes y_2^{op} \rangle,$$

from which the result follows. \square

From this we can deduce the following corollary, which is due to Ocneanu [13] in the finite-depth case, and Popa [16] in the general case.

Corollary 3.6. *We have*

$${}_{M_0 \otimes M_0^{op}} L^2(M_0 \boxtimes M_0)_{M_0 \otimes M_0^{op}} \simeq \bigoplus_{v \in \Gamma_+} X_v \otimes \overline{X_v}.$$

Proof. Let Ξ_k denote the range of Ψ_k , which is a closed subspace of $L^2(M_0 \boxtimes M_0)$ invariant under the left and right actions of $M_0 \otimes M_0^{op}$. By Theorem 3.5, Ξ_k is isomorphic to

$$\bigoplus_{v \in \Gamma_{+,k}} X_v \otimes \overline{X_v}$$

as a $M_0 \otimes M_0^{op}$ -bimodule. It follows that $\overline{\bigcup_{k \geq 0} \Xi_k}$ is isomorphic as a $M_0 \otimes M_0^{op}$ -bimodule to

$$\bigoplus_{v \in \Gamma_+} X_v \otimes \overline{X_v}.$$

Now if $x \in V_{0,0}(2s, 2t)$ then $x = \Psi_t(x_t)$, where $x_t \in M_t \otimes M_t^{op}$ is defined by

$$x_t = \sum_{v \in \Gamma_{+,t}} \sqrt{\frac{n_k(v)}{t_k(v)}} \cdot \begin{array}{c} \boxed{x} \\ \hline \boxed{q_t(v)} \star \end{array}$$

So $\bigcup_{k \geq 0} \Xi_k$ contains $Gr_0 \boxtimes Gr_0$ and is therefore dense in $L^2(M_0 \boxtimes M_0)$, which completes the proof. \square

Remark 3.7. Note that it follows from the proof of Corollary 3.6 that Ψ_k is surjective if the depth of \mathcal{P} is less than or equal to $2k$, i.e. if $\Gamma_+ = \Gamma_{+,k}$.

Corollary 3.8. *Let $k \geq 0$, then if Γ_+ is finite the index $[M_k \boxtimes M_k : M_k \otimes M_k^{op}]$ is equal to the global index*

$$I = \sum_{v \in \Gamma_+} \dim_{M_0}(X_v)^2.$$

Otherwise $[M_k \boxtimes M_k : M_k \otimes M_k^{op}] = \infty$.

Proof. For $k = 0$ this is clear from the previous corollary. The result then follows for $k \geq 1$ from Lemma 3.2 and the fact that the global index of \mathcal{P}^{op} (the planar algebra obtained from \mathcal{P} by reversing shading) is the same as that of \mathcal{P} . \square

4. DERIVATIONS ON PLANAR ALGEBRA FACTORS

Throughout this section we will denote $A = Gr_0(\mathcal{P})$ and $M = M_0(\mathcal{P})$, where \mathcal{P} is a subfactor planar algebra. Define

$$\Phi_o = \bigoplus_{s,t \geq 0} V_{1,0}^-(2s+1, 2t), \quad \Phi_e = \bigoplus_{s,t \geq 0} V_{1,0}^+(2s, 2t+1), \quad \Phi = \Phi_o \oplus \Phi_e.$$

Observe that Φ has a natural right $A \otimes A^{op}$ -module structure.

Given $Q \in \Phi_o$ and $x \in P_n$, define $\tilde{\delta}_Q(x) \in A \boxtimes A$ by

$$\tilde{\delta}_Q(x) = \sum_{\substack{0 \leq k \leq 2n \\ k \text{ even}}} \text{Diagram: A box labeled } x \text{ connected to a box labeled } Q \text{ via a wire with a loop labeled } k.$$

Extend $\tilde{\delta}_Q$ linearly to obtain a map $\tilde{\delta}_Q : A \rightarrow A \boxtimes A$. If $Q \in \Phi_e$ we define $\tilde{\delta}_Q : A \rightarrow A \boxtimes A$ likewise, except that we sum over odd instead of even k in the formula above. We then extend $Q \mapsto \tilde{\delta}_Q$ linearly to Φ .

Proposition 4.1. *If $Q \in \Phi$, then $\tilde{\delta}_Q$ is a derivation from A into $A \boxtimes A$. The assignment $Q \mapsto \tilde{\delta}_Q$ is a $A \otimes A^{op}$ -linear map from Φ into $\text{Der}(A, A \boxtimes A)$.*

Proof. For the first statement, it suffices to consider $Q \in \Phi_o$ or $Q \in \Phi_e$. In either case the result is clear from drawing the relevant diagram. The second statement is obvious. \square

Let $E_{M \otimes M^{op}}$ denote the unique trace-preserving conditional expectation from $M \boxtimes M$ onto $M \otimes M^{op}$, which is given by the orthogonal projection $L^2(M \boxtimes M) \rightarrow L^2(M \otimes M^{op})$. We will now show that this projection may also be computed with respect to the orthogonal basis of Section 2.1. Let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the inner product on $A \boxtimes A$ induced by δ^{-2k} times the restriction of Tr' .

Lemma 4.2. *Let P denote the orthogonal projection of $A \boxtimes A$ onto $A \otimes A^{op}$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. If $Q \in A \boxtimes A$ then*

$$E_{M \otimes M^{op}}(Q) = P(Q).$$

Proof. If $Q \in A \boxtimes A$ and $a \otimes b \in A \otimes A^{op}$ we have

$$\langle Q, a \otimes b \rangle = \text{Diagram: Box } Q \text{ connected to a stack of boxes } \sum TL, a^*, b^*, \sum TL. = \text{Diagram: Box } P(Q) \text{ connected to the same stack.} = \langle P(Q), a \otimes b \rangle,$$

so that $P(Q)$ is equal to the orthogonal projection of Q onto $M \otimes M^{op}$ in $L^2(\tau_k \boxtimes \tau_k)$. \square

Corollary 4.3.

- (1) $E_{M \otimes M^{op}}(V_{0,0}^+(2s, 2t)) \subset P_s \otimes P_t$, in particular $E_{M \otimes M^{op}}(A \boxtimes A) \subset A \otimes A^{op}$.
- (2) Suppose that $Q \in A \boxtimes A \cap (A \otimes A^{op})^\perp$. Write

$$Q = \sum_{s,t \geq 0} Q_{s,t},$$

where $Q_{s,t} \in V_{0,0}^+(2s, 2t)$ are nonzero for finitely many $s, t \geq 0$. Then $Q_{s,t} \in (A \otimes A^{op})^\perp$ for all $s, t \geq 0$.

- (3) If $Q \in V_{0,0}^+(2s, 2t) \cap (A \otimes A^{op})^\perp$, then

$$\text{Diagram: Box } Q \text{ with a loop.} = 0.$$

Proof. (1) and (2) are immediate from the previous lemma. Let Q be as in (3), then if $x \in P_t$ we have

$$\text{Diagram: Box } Q \text{ connected to box } x. = \text{Diagram: Box } P(Q) \text{ connected to box } x. = 0,$$

from which the result follows. \square

Given $Q \in \Phi$, define $\delta_Q = E_{M \otimes M^{op}} \circ \tilde{\delta}_Q$. Note that the range of δ_Q is contained in $A \otimes A^{op}$ by the previous corollary. Since $E_{M \otimes M^{op}}$ commutes with the left and right actions of $A \otimes A^{op}$, it is clear that $\delta_Q \in \text{Der}(A, A \otimes A^{op})$ and that the assignment $Q \mapsto \delta_Q$ is $A \otimes A^{op}$ -linear.

We will now determine the kernel of the mapping $Q \mapsto \delta_Q$. Define

$$\Omega_o = \bigoplus_{s,t \geq 0} V_{0,0}^-(2s+1, 2t+1), \quad \Omega_e = \bigoplus_{s,t \geq 0} V_{0,0}^+(2s, 2t), \quad \Omega = \Omega_o \oplus \Omega_e.$$

Observe that $\Omega_e = A \boxtimes A$, in particular $A \otimes A^{op} \subset \Omega$. The orthogonal complement of $A \otimes A^{op}$ in Ω is $\Omega_o \oplus (A \otimes A^{op})^\perp \cap A \boxtimes A$.

Given $R \in \Omega_o$, define $\rho(R) \in \Phi$ by

$$\rho(R) = \text{diagram of } R \text{ with top-left and bottom-right corners shaded} - \text{diagram of } R \text{ with top-right and bottom-left corners shaded}$$

Likewise if $R \in \Omega_e$ we define $\rho(R) \in \Phi$ by the same formula, but note that the shading of the “corners” must be reversed. We then extend ρ to Ω linearly.

Proposition 4.4. *The restriction of ρ to $\Omega \cap (A \otimes A^{op})^\perp$ is an $A \otimes A^{op}$ -linear isomorphism onto the kernel of the mapping $Q \mapsto \delta_Q$.*

Proof. If $R \in \Omega_o$, then it is immediate from the defining formula that $\tilde{\delta}_{\rho(R)} = 0$. On the other hand, if $R \in \Omega_e$ then it is easy to see that $\tilde{\delta}_{\rho(R)}(x) = [x, R]$, and therefore $\delta_{\rho(R)}(x) = [x, E_{M \otimes M^{op}}(R)]$. So we have $\delta_{\rho(R)} = 0$ if $R \perp A \otimes A^{op}$. In fact since M is diffuse, any Hilbert-Schmidt operator commuting with A must be zero, and so we have $\delta_{\rho(R)} = 0$ if and only if $R \perp A \otimes A^{op}$.

It remains to show that if $Q \in \Phi$ is such that $\delta_Q = 0$, then $Q = \rho(R)$ for a unique $R \in \Omega$. Write

$$Q = \sum_{s,t \geq 0} Q_{s,t}$$

where $Q_{s,t} \in V_{1,0}^{(-1)^s}(s, t)$ are nonzero for finitely many s, t . For $s, t \geq 1$ with $s+t$ even, define $R_{s,t} \in V_{0,0}^{(-1)^s}(s, t)$ by

$$R_{s,t} = \sum_{k=1}^{\min\{s,t+1\}} \text{diagram of } R_{s,t} \text{ with } k-1 \text{ on the left and } Q_{s+k,t+1-k} \text{ on the right}$$

Note that $R_{s,t} = 0$ for all but finitely many s, t , and set $R = \sum_{s,t} R_{s,t} \in \Omega$. We claim that $Q = \rho(R)$, note that it will then follow from the discussion above that $R \in (A \otimes A^{op})^\perp$.

We need to show that for any $s, t \geq 0$ we have

$$Q_{s,t} = \text{diagram of } Q_{s,t} \text{ with top-left and bottom-right corners shaded} - \text{diagram of } Q_{s,t} \text{ with top-right and bottom-left corners shaded}$$

We will prove this by induction on s . First suppose that $s = 0$, we claim that $Q_{s,t} = 0$ for any $t \geq 0$. Indeed, this follows from $\delta_Q(\cup) = 0$ and (2) of Corollary 4.3.

So assume that $s \geq 1$. Let x be the following Temperley-Lieb element with $2(s+1)$ boundary points:

$$x = \text{diagram of } x \text{ with } s \text{ cups and } s \text{ caps}$$

Since $\delta_Q(x) = 0$, applying (2) of Corollary 4.3 we find that

$$\sum_{k=1}^s \left| \text{Diagram 1} \right| + \sum_{k=0}^{\min\{s,t\}} \left| \text{Diagram 2} \right| \in (A \otimes A^{op})^\perp,$$

where the number of parallel lines appearing at left is determined by the condition that each term is in $V_{0,0}^+(2s, t+s+1)$.

Applying induction to the first sum and simplifying, we are left with

$$\sum_{k=0}^{\min\{s,t\}} \left| \text{Diagram 3} \right| - \left| \text{Diagram 4} \right| \in (A \otimes A^{op})^\perp.$$

By (3) of Corollary 4.3, we have

$$\left| \text{Diagram 5} \right| + \sum_{k=1}^{\min\{s,t\}} \left| \text{Diagram 6} \right| - \left| \text{Diagram 7} \right| = 0.$$

After rotating s strings clockwise from bottom to top, and one more from the bottom to the left, we obtain the desired result.

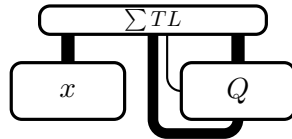
Finally, if $Q = \rho(R')$ and R is defined in terms of Q as above, it is straightforward to check that $R = R'$, which proves that ρ is injective. Since ρ is clearly linear over $A \otimes A^{op}$, this completes the proof. \square

We now compute the “conjugate variable” $\delta_Q^*(1 \otimes 1)$. This is closely related to the Schwinger-Dyson equation from [9, Lemma 12].

Proposition 4.5. *Let $Q \in V_{1,0}^{(-1)^s}(s,t)$, and view δ_Q as a densely defined unbounded operator $\delta_Q : L^2(M) \rightarrow L^2(M \otimes M^{op})$. Then $1 \otimes 1 \in \mathfrak{D}(\delta_Q^*)$, and*

$$(\delta_Q^*(1 \otimes 1))^* = \left| \text{Diagram 8} \right| - \sum_{\substack{0 \leq k \leq s \\ k+t \in 2\mathbb{Z}}} \left| \text{Diagram 9} \right| - \sum_{\substack{0 \leq k \leq t \\ k+s \in 2\mathbb{Z}}} \left| \text{Diagram 10} \right|$$

Proof. Fix $x \in A$ and look at



Consider the string on the left side of Q , for a fixed term in the sum over TL diagrams there are three possibilities: this string is connected to the top of Q , to the bottom of Q , or to x . It follows that the inner product of x with the element in the statement of the proposition is equal to the sum of the terms above in which the string on the left side of Q is connected to x . But it is clear that this is equal to $(\tau_k \boxtimes \tau_k) \tilde{\delta}_Q(x) = \langle \delta_Q(x), 1 \otimes 1 \rangle$, which completes the proof. \square

Corollary 4.6. *If $Q \in \Phi$ then δ_Q is closable as an unbounded operator $L^2(M) \rightarrow L^2(M \otimes M^{op})$.*

Proof. This follows from $1 \otimes 1 \in \mathfrak{D}(\delta_Q^*)$ by [20]. \square

Note that Φ is identified with $(p_{1,+} + p_{1,-})Vp_{0,+}$, and so may be completed to obtain a Hilbert space $\overline{\Phi} \simeq (p_{1,+} + p_{1,-})L^2(\mathfrak{M})p_{0,+}$. Likewise Ω may be completed to obtain a Hilbert space $\overline{\Omega} \simeq (p_{0,+} + p_{0,-})L^2(\mathfrak{M})p_{0,+}$.

Proposition 4.7. *As right $M \otimes M^{op}$ modules, we have*

$$\dim_{M \otimes M^{op}}(\overline{\Phi}) = 2\delta I,$$

and

$$\dim_{M \otimes M^{op}}(\overline{\Omega}) = 2I.$$

Proof. Let $c \in V_{0,1}^-(1, 0)$ be the following element:

$$c = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$$

It follows from the same argument as in Lemma 3.2 that $L^2(M \boxtimes M) \simeq q \cdot \overline{\Phi_o}$ as right $M \otimes M^{op}$ -modules, where $q = c^*(cc^*)^{-1}c$ is the initial projection of c^*c . Note that there is an obvious inclusion of M_1 into the commutant of the right action of $M \otimes M^{op}$ on $\overline{\Phi_o}$. It follows from the uniqueness of the trace on M_1 that the trace of q in the commutant of $M \otimes M^{op}$ is equal to its trace in M_1 , which is δ^{-1} . By Corollary 3.8 we then have

$$\dim_{M \otimes M^{op}} \overline{\Phi_o} = \delta \cdot I.$$

Likewise we have $\dim_{M \otimes M^{op}} \overline{\Phi_e} = \delta \cdot I$, so that $\dim_{M \otimes M^{op}} \overline{\Phi} = 2\delta I$.

Clearly we have $\overline{\Omega_e} \simeq L^2(M \boxtimes M^{op})$, and hence $\dim_{M \otimes M^{op}} \overline{\Omega_e} = I$. On the other hand, let $c' \in V_{0,1}^+(1, 0)$ be as c above but with the shading reversed, and set $q' = c'^*(cc'^*)^{-1}c'$. Then by the same argument as Lemma 3.2, we have $\overline{\Omega_o} \simeq q' \cdot \overline{\Phi_e}$, and hence $\dim_{M \otimes M^{op}} \overline{\Omega_o} = I$. \square

Remark 4.8. For a family X_1, \dots, X_n of self-adjoint operators which generate a II_1 factor M , Voiculescu has defined the *free entropy dimension* $\delta_0(X_1, \dots, X_n)$. If M is isomorphic to an interpolated free group factor $L\mathbb{F}_t$, it is generally expected (but not proven) that $\delta_0(X_1, \dots, X_n) = t$.

Beginning with the work of Connes-Shlyakhtenko on L^2 -Betti numbers [2], there have been a number of recent results relating Voiculescu's free entropy dimension to the Murray von Neumann dimension of certain spaces of derivations. In particular, it is reasonable to expect that in "nice" situations, we should have the following equality:

$$\dim_{M \otimes M^{op}} \overline{\text{Der}_c(A, A \otimes A^{op})}^{(L^2(M \otimes M^{op}))^n} = \delta_0(X_1, \dots, X_n),$$

where $A = \mathbb{C}\langle X_1, \dots, X_n \rangle$ and $\text{Der}_c(A, A \otimes A^{op})$ denotes the module of closable derivations $\delta : A \rightarrow A \otimes A^{op}$, which is embedded into $(L^2(M \otimes M^{op}))^n$ via $\delta \mapsto (\delta(X_1), \dots, \delta(X_n))$. Note in particular that if X_1, \dots, X_n are free semicircular random variables, then both sides of the equation are indeed equal to n .

Combining the results above, we have constructed a $A \otimes A^{op}$ -linear map from $D = \{\delta_Q : Q \in \Phi\}$ onto a dense subset of the orthogonal complement in $\overline{\Phi}$ of $\rho(\overline{\Omega})$. It follows that

$$\dim_{M \otimes M^{op}} \overline{D}^{\overline{\Phi}} = \dim_{M \otimes M^{op}} \overline{\Phi} - \dim_{M \otimes M^{op}} \rho(\overline{\Omega} \ominus L^2(M \otimes M^{op})).$$

One can show that the extension of ρ to $\overline{\Omega}$ is still injective, but the proof is rather tedious and so we omit it. Assuming this, we are led to the equation

$$\dim_{M \otimes M^{op}} \overline{D}^{\overline{\Phi}} = 2\delta I - (2I - 1) = 1 + 2I(\delta - 1) = r_0.$$

Now if we have $D = \text{Der}_c(A, A \otimes A^{op})$, then in light of the remark above this "explains" the formula for r_0 from [8]. Similar reasoning may of course be applied to the parameters r_k , $k \geq 1$.

So we are led to the question of whether every closable derivation $\delta : A \rightarrow A \otimes A^{op}$ is equal to δ_Q for some $Q \in \Phi$. First note that it is not the case that every (algebraic) derivation $A \rightarrow A \otimes A^{op}$ is of this form. For example, when \mathcal{P} is the tensor planar algebra on \mathbb{C}^n then $A \simeq \mathbb{C}\langle t_1, \dots, t_{n^2} \rangle$ is the free

$$\begin{aligned} \widetilde{\delta}\left(\textcircled{2n}\right) &= \sum_{\substack{s,t \\ s \text{ even} \\ t \text{ odd}}} \sum_{k=1}^n \text{Diagram 1} + \sum_{\substack{s,t \\ s \text{ odd} \\ t \text{ even}}} \sum_{k=0}^{n-1} \text{Diagram 2} \\ \widetilde{\delta}\left(\textcircled{2n}\right) &= \sum_{\substack{s,t \\ s \text{ odd} \\ t \text{ even}}} \sum_{k=1}^n \text{Diagram 3} + \sum_{\substack{s,t \\ s \text{ even} \\ t \text{ odd}}} \sum_{k=0}^{n-1} \text{Diagram 4} \end{aligned}$$
$$\tilde{\delta}(y) = \tilde{\delta}\left(\begin{array}{c} \text{---} p \text{---} \\ | \\ \boxed{y} \\ | \\ \text{---} q \text{---} \end{array}\right),$$
$$R_{s,t} = \sum_{k=1}^s \text{Diagram 1} + \sum_{k=0}^s \text{Diagram 2} + \sum_{k=1}^{p/2} \text{Diagram 3} + \sum_{k=1}^{p/2} \text{Diagram 4}$$

On the other hand, we also have

$$\tilde{\delta}(y) = \tilde{\delta} \left(\begin{array}{c} \blacksquare \\ p+1 \end{array} \quad y \quad \begin{array}{c} \blacksquare \\ q-1 \end{array} \right).$$

It then follows by similar computations that

$$R_{s,t} = \sum_{k=1}^s \text{Diagram 1} + \sum_{k=0}^{s-1} \text{Diagram 2} + \sum_{k=0}^{p/2} \text{Diagram 3} + \sum_{k=1}^{p/2} \text{Diagram 4}$$

Setting these two expressions for $R_{s,t}$ equal to each other and cancelling out common terms, we obtain the desired formula.

By the same argument, if $p, q \geq 1$ are odd and $p + q = 2n$, then for any $y \in P_n$ and any odd s and even t we have:

$$\star \text{Diagram 1} = \text{Diagram 2} \star$$

Now for $s \geq 1$ odd and $t \geq 0$ even, define $Q_{s,t} \in V_{1,0}^-(s, t)$ by

$$Q_{s,t} = \text{Diagram 1} \cdot \text{Diagram 2}$$

We claim that

$$\text{Diagram 1} = E_{M \otimes M^{op}} \left[\text{Diagram 2} \right]$$

Indeed, using the relation above we see that for $x \in V_{0,2n+1}(s, 0)$ and $y \in V_{t,0}(2n, 0)$ we have:

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}$$

The result then follows from Lemma 4.2.

Likewise, for $s \geq 0$ even and $t \geq 1$ odd, define

$$Q_{s,t} = \text{Diagram 1} \cdot \text{Diagram 2}$$

A similar argument shows that

$$\begin{array}{c} \text{---} \\ | \\ \boxed{a_{2n,s}} \\ | \\ \boxed{b_{2n+1,t}} \\ | \\ \text{---} \end{array} = E_{M \otimes M^{op}} \left[\begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \\ | \\ \boxed{Q_{s,t}} \\ | \\ | \\ | \\ | \\ \text{---} \end{array} \right]$$

Now set $Q = \sum_{s,t} Q_{s,t}$, we claim that $\delta = \delta_Q$, where δ is the restriction of $\tilde{\delta}$ to $A = e_0 \mathcal{V}_+ e_0$. Indeed, for $x \in P_n$ we have

$$\begin{aligned} \delta(x) &= \tilde{\delta} \left(\begin{array}{c} \text{---} \\ | \\ \boxed{\text{---}} \\ | \\ \text{---} \end{array} \right) = \sum_{\substack{s,t \\ s \text{ odd} \\ t \text{ even}}} \sum_{k=1}^n \begin{array}{c} \boxed{a_{2(n-k)+1,s}} \\ | \\ \boxed{b_{2(n-k),t}} \end{array} + \sum_{\substack{s,t \\ s \text{ even} \\ t \text{ odd}}} \sum_{k=0}^{n-1} \begin{array}{c} \boxed{a_{2(n-k),s}} \\ | \\ \boxed{b_{2(n-k)+1,t}} \end{array} \\ &= \sum_{\substack{s,t \\ s \text{ odd} \\ t \text{ even}}} \sum_{k=1}^n E_{M \otimes M^{op}} \left[\begin{array}{c} \boxed{a_{2(n-k)+1,s}} \\ | \\ \boxed{Q_{s,t}} \end{array} \right] + \sum_{\substack{s,t \\ s \text{ even} \\ t \text{ odd}}} \sum_{k=0}^{n-1} E_{M \otimes M^{op}} \left[\begin{array}{c} \boxed{a_{2(n-k),s}} \\ | \\ \boxed{Q_{s,t}} \end{array} \right] = \delta_Q(x), \end{aligned}$$

which completes the proof. \square

Remark 4.10. As mentioned above, we expect that every closable derivation $\delta : A \rightarrow A \otimes A^{op}$ has an extension to a derivation $\tilde{\delta} : \mathcal{V}_+ \rightarrow \mathcal{V}_+ e_0 \otimes e_0 \mathcal{V}_+^{op}$ which vanishes on \mathcal{A}_+ . To show this it would be enough to find a system matrix units $\{e_{ij}\}$ in the completion of \mathcal{V}_+ such that $e_{11} = e_0$, and which are smooth enough that $e_{1i} \mathcal{V}_+ e_{j1}$ is contained in a holomorphic closure of A . Once we have such a system, we can define $\tilde{\delta}$ by

$$\tilde{\delta}(a) = \sum_{i,j} e_{i1} \bar{\delta}(e_{1i} a e_{j1}) e_{1j},$$

where $\bar{\delta}$ denotes the closure and we use crucially that $e_{1i} a e_{j1} \in \mathfrak{D}(\bar{\delta})$ for any $a \in \mathcal{V}_+$.

The existence of such a system of matrix units should in principle follow from a more careful analysis of the arguments in [8], and of the work of Dykema [4], [5]. Since we are only aiming in this section to provide some intuition for the parameter r_0 , and its relation to derivations on A , we will avoid such technical questions. But let us point out that we do have natural extensions of the derivations δ_Q to \mathcal{V}_+ as follows.

Proposition 4.11. *If $Q \in \Phi$, then there is a derivation $\tilde{\delta} : \mathcal{V}_+ \rightarrow \mathcal{V}_+ e_0 \otimes e_0 \mathcal{V}_+^{op}$ such that $\tilde{\delta}|_{\mathcal{A}_+} = 0$ and $\tilde{\delta}|_A = \delta_Q$.*

Proof. Let $Q = \sum_{s,t \geq 0} Q_{s,t}$, where $Q_{s,t} \in V_{1,0}^{(-1)^s}(s,t)$. Define $\tilde{\delta}$ by $\tilde{\delta}|_{\mathcal{A}_+} = 0$ and

$$\begin{aligned} \tilde{\delta}(c_n) &= \sum_{\substack{s \text{ even} \\ t \text{ odd}}} a_{2n,s} \otimes b_{2n+1,t}^{op}, \\ \tilde{\delta}(c_n^*) &= \sum_{\substack{s \text{ odd} \\ t \text{ even}}} a_{2n+1,s} \otimes b_{2n,t}^{op}, \end{aligned}$$

where $a_{m,s} \otimes b_{m,t}^{op} \in V_{m,0}^{(-1)^s}(s,0) \otimes V_{0,m}^+(t,0)$ are defined by

$$\begin{aligned} \begin{array}{c} \text{---} \boxed{a_{2n+1,s}} \\ \text{---} \boxed{b_{2n,t}} \end{array} &= E_{M_0 \otimes M_0^{op}} \left[\begin{array}{c} \text{---} \boxed{2n} \\ \text{---} \boxed{Q_{s,t}} \end{array} \right], \\ \begin{array}{c} \text{---} \boxed{a_{2n,s}} \\ \text{---} \boxed{b_{2n+1,t}} \end{array} &= E_{M_0 \otimes M_0^{op}} \left[\begin{array}{c} \text{---} \boxed{2n-1} \\ \text{---} \boxed{Q_{s,t}} \end{array} \right]. \end{aligned}$$

It then follows from the proof of Proposition 4.9 that $\tilde{\delta}$ is a well-defined derivation, and that its restriction to A is δ_Q . \square

Remark 4.12. In Voiculescu's free analysis [21],[22], the central objects are derivations $\delta : A \rightarrow A \otimes A$ which are *coassociative*, i.e.

$$(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta) \circ \delta.$$

Unfortunately, there do not appear to be any natural coassociative derivations among the δ_Q which we have considered in this section.

There is however a natural way of defining coassociative derivations on A as follows. Given $Q \in P_1$, define $\partial_Q : A \rightarrow A \otimes A$ by

$$\partial_Q(x) = \sum_{k=0}^n (\text{id} \otimes \text{op}) E_{M \otimes M^{op}} \left[\begin{array}{c} \text{---} \boxed{x} \text{---} \boxed{2k} \text{---} \boxed{Q} \end{array} \right]$$

It is easy to see that ∂_Q is coassociative, so that (A, ∂_Q) forms a (graded) *infinitesimal bialgebra* [21],[1]. However these derivations do not behave well with respect to the Voiculescu trace, in particular they do not appear to be closable.

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